The Picard functors

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1 Introduction

In this report, we introduce relative Picard functors with respect to different sheafifications. The main goal is to compare them under certain assumptions. In order to do this, we need the tool of rigidifications. In the end, we loose a bit on our assumptions, and accordingly come to its generalization, rigidificators.

2 Notations and conventions

Let us first fix some notations throughout this whole report:

(Sch/X) denotes the category of schemes over a fixed scheme X with X-morphisms of schemes, (Sets) denotes the category of sets and maps of sets, (Ab) the category of abelian groups and group morphisms between them.

Given a base scheme S, $f : X \to S$ is called *a scheme over* S, the morphism f is called the *structure morphism*. Denote by $f_T : X \times_S T \to T$ the base change with respect to $T \in (\mathrm{Sch}/S)$. When saying a functor we always mean a covariant functor $F : (\mathrm{Sch}/S)^{\mathrm{opp}} \to (\mathrm{Sets})$.

By saying a diagram of sets $A \xrightarrow{f} B \underset{h}{\overset{g}{\rightrightarrows}} C$ is exact, we mean that f maps bijectively to the set $\{\beta \in B \mid g(\beta) = h(\beta)\}$.

3 Basics facts from algebraic geometry

Definition 3.1. (**Site**, [Sta] Tag/03NF) *A site* consists of a category $\mathcal{C}$ and a set $\mathfrak{M}$ consisting of families of morphisms with fixed target *i.e.* , the assignment to each $U \in \mathrm{Obj}(\mathcal{C})$ of a collection of sets of arrows $\{U_i \to U\}_{i \in I}$, called *coverings*, so that the following conditions are satisfied:

(i) $\mathfrak{M}$ contains all isomorphisms: if $\varphi : V \to U$ is an isomorphism in $\mathcal{C}$, then $\{\varphi : V \to U\}$ is a covering,

(ii) Fiber products exist and $\mathfrak{M}$ is stable under fiber product: if $\{U_i \to U\}_{i \in I}$ is a covering and $V \to U$ is a morphism in $\mathcal{C}$, then

  (a) for all $i \in I$ the fibre product $U_i \times_U V$ exists in $\mathcal{C}$;

  (b) $\{U_i \times_U V \to V\}_{i \in I}$ is a covering.

(iii) $\mathfrak{M}$ is stable under composition: if $\{\varphi_i : U_i \to U\}_{i \in I}$ is a covering and for all $i \in I$ we are given a covering $\{\psi_{ij} : U_{ij} \to U_I\}$, then $\{\varphi_i \circ \psi_{ij} : U_{ij} \to U\}_{(i,j) \in \prod_{i \in I}\{i\} \times I_i}$ is also a covering.

It is assumed that its collection of objects form a set and collection of coverings of a site is a set as well. What we call a site is also called *a category endowed with a pretopology* by [SGA4II] Définition 1.3.

We now introduce some frequently used definitions of sites following [BLR] section 8.1 and [MilLEC]. Let S be a given scheme, and let $X \in (\mathrm{Sch}/S)$ be a fixed scheme for the following discussion.

(1) X_{fpqc} is the site whose underlying category is (Sch/X), whose covering $\{\varphi_i : U_i \to U\}_{i \in I}$ is defined as any jointly surjective collection of maps such that each φ is flat and for each affine open $V \to U$, there exists a finite set K, a map $i : K \to I$ and affine open $V_{i(k)} \subset U_{i(k)}$ such that $V = \bigcup_{k \in K} \varphi_{i(k)}(V_{i(k)})$.

(2) X_{fppf} is the site whose underlying category is (Sch/X), whose covering is any jointly surjective collection of flat and locally of finite presentation maps $\{U_i \to U\}_{i \in I}$.

(3) $X_{\text{ét}}$ is the site whose underlying category is $U \to X$ étale schemes over X, a subcategory of (Sch/X), whose covering is any $\{U_i \to U\}$ jointly surjective collection of étale maps, locally of finite presentation.

(4) X_{Zar} is the site whose underlying category is (Sch/X), whose covering is any jointly surjective collection of open immersions $\{U_i \to U\}$.

We write $\mathrm{H}^i(X_{\mathrm{top}}, F)$ for the i-th sheaf cohomology group taken with respect to the top-site, with $\mathrm{top} = \text{ét},\ \mathrm{fppf},\ \mathrm{fpqc},\ \mathrm{Zar}$. If we do not write any sub-index, we consider Zariski topology.

Definition 3.2. A functor F is called a *sheaf with respect to* $\mathfrak{M}$ (or $\mathfrak{M}$-sheaf) if for all morphisms $T' \to T$ in $\mathfrak{M}$,

(1) $F(\coprod T_i) \to \prod F(T_i)$ is an isomorphism, for $(T_i)_{i\in I}$ any family of S-schemes.

(2) $F(T) \to F(T') \rightrightarrows F(T' \times_T T')$ is exact.

In practice, we often take $T' = \coprod T_i \to T$ for a covering $\{T_i\}$.

Theorem 3.3. *(**Grothendieck**) Let F be a representable functor from* (Sch/S) *to* (Sets)*, then F is a sheaf with respect to $\mathfrak{M}_{\mathrm{fpqc}}$.*

Proof. See [BLR] proposition 1 or [FGA Explained] part I theorem 2.55. □

With this theorem, it is natural for us to ask whether a functor is a $\mathfrak{M}_{\mathrm{fpqc}}$-sheaf first if we want to know if it is representable.

Example 3.4. For $Z \in (\mathrm{Sch}/S)$, the functor $\mathbb{G}_m : Z \mapsto \mathrm{H}^0(Z, \mathcal{O}_Z^*)$ is a sheaf for all four topologies mentioned above: it is represented by the S-scheme $\mathrm{Spec}(\mathcal{O}_S[u, u^{-1}])$ for u indeterminant.

4 Relative Picard functor

Definition 4.1. Define $\mathrm{Pic}(X)$ to be the group of isomorphism classes of invertible sheaves on X, call it *the absolute Picard group* of X (with respect to the Zariski topology).

Proposition 4.2. $\mathrm{Pic}(X) \cong \mathrm{H}^1(X_{\acute{e}t}, \mathbb{G}_m) \cong \mathrm{H}^1(X_{\mathrm{Zar}}, \mathbb{G}_m) \cong \mathrm{H}^1(X_{\mathrm{fppf}}, \mathbb{G}_m) \cong \mathrm{H}^1(X_{\mathrm{fpqc}}, \mathbb{G}_m)$.

Proof. [Ha83] Ex. III, 4.5, p. 224 for Zariski case. Others see [FGA] p. 190-16 or [MilLEC] Theorem 11.4. The idea is that descent with respect to fpqc-morphisms turns line bundles to line bundles. □

The functor $P_{X/S}$ defined by map $T \mapsto \mathrm{Pic}(X \times_S T)$ is not a sheaf, thus by theorem 3.3 it is not representable. We need to do a sheafification process with respect to $\mathfrak{M}_{\mathrm{fppf}}$ and $\mathfrak{M}_{\mathrm{Zar}}$, as in [BLR] page 201.

Definition 4.3. The fppf-sheaf associated to the functor

$$P_{X/S} : (\mathrm{Sch}/S)^{\mathrm{opp}}_{\mathrm{fppf}} \to (\mathrm{Sets}), \quad T \mapsto \mathrm{Pic}(X \times_S T)$$

is called *the relative Picard functor* of X over S. It is denoted by $\mathrm{Pic}_{X/S,\mathrm{fppf}}$, or simply $\mathrm{Pic}_{X/S}$ if no confusion is caused. For any $T \in (\mathrm{Sch}/S)$, we call $\mathrm{Pic}_{X/S}(T)$ the *relative Picard group* of $X \times_S T$ over T.

We know $P_{X/S}$ is actually a functor to (Ab) with group operation given by tensor product. Similarly, we denote by different sub-indexes the sheafification with respect to different Grothendieck topologies.

From the sheafification process, we know that for every $T \in (\mathrm{Sch}/S)$, each element of $\mathrm{Pic}_{X/S,\mathrm{fppf}}(T)$ is represented by an invertible sheaf $\mathcal{L}'$ on $X_{T'}$ for some fppf covering $T' \to T$. Moreover, there must be an fppf-covering $\bar{T} \to T' \times_T T'$ such that the two pullbacks of $\mathcal{L}'$ to $X_{\bar{T}}$ are isomorphic.

Furthermore, another such sheaf $\mathcal{L}_1$ on X_{T_1} represents the same $\mathrm{Pic}_{X/S,\mathrm{fppf}}(T)$ if and only if there is an fppf-covering $T_1' \to T_1 \times_T T'$ such that the pullbacks of $\mathcal{L}'$ and $\mathcal{L}_1$ to $X_{T_1'}$ are isomorphic. Of course, similar considerations apply to the sheafifications with respect to the Zariski, étale, and fpqc topologies as well.

Proposition 4.4. *For $T \in (\mathrm{Sch}/S)$, we have $\mathrm{Pic}_{X/S,\mathrm{top}}(T) = \mathrm{H}^0(T_{\mathrm{top}}, \mathrm{R}^1 f_{T*}\mathbb{G}_m)$ and* top *equals to* Zar, fppf*, and ét.*

Proof. All proofs are similar to the Zariski case, see [Ha83] III, 8.1. □

Proposition 4.5. *For $T \in (\mathrm{Sch}/S)$, $p \in \mathbb{N}$, the canonical morphisms from the $\mathrm{H}^p(T_{\acute{e}t}, \mathbb{G}_m)$ to $\mathrm{H}^p(T_{\mathrm{fppf}}, \mathbb{G}_m)$ induced by morphism of sites ([MilLEC] definition 5.2 p40) are isomorphisms.*

Proof. [Dix] p. 180. □

The following proposition is our first result on comparison of different relative Picard functors, and we get some useful byproducts in its proof.

Proposition 4.6. *Let $f : X \to S$ be a quasi-compact and quasi-separated morphism. Assume that $f_*(\mathcal{O}_X) = \mathcal{O}_S$ (resp. $f_*(\mathcal{O}_X) = \mathcal{O}_S$ holds universally,* i.e. *$f_*(\mathcal{O}_{X_T}) = \mathcal{O}_T$ holds for any base change $T \to S$), then if $T \in (\mathrm{Sch}/S)$ is flat over S (*resp. *for each $T \in (\mathrm{Sch}/S)$), the canonical sequence*

$$0 \longrightarrow \mathrm{Pic}(T) \to \mathrm{Pic}(X \times_S T) \longrightarrow \mathrm{Pic}_{X/S}(T) \longrightarrow \mathrm{H}^2(T_{\acute{e}t}, f_{T*}(\mathbb{G}_m)) \longrightarrow \mathrm{H}^2(X_{T,\acute{e}t}, \mathbb{G}_m)$$

is exact. If f admits a section, then the sequence

$$0 \longrightarrow \mathrm{Pic}(T) \longrightarrow \mathrm{Pic}(X \times_S T) \longrightarrow \mathrm{Pic}_{X/S}(T) \longrightarrow 0$$

is exact.

Proof. For any $T \in (\mathrm{Sch}/S)$, we assume one of the following holds: (1) $f_*(\mathcal{O}_X) = \mathcal{O}_S$ holds universally. (2) T is flat over S. The following process is valid for top- chosen as Zar-, ét- or fppf-. The Leray spectral sequence (see [SGA4II] Exp.V. §3) associated to $X_{T,\mathrm{top}} \to T_{\mathrm{top}}$ and to the sheaf $\mathbb{G}_m$ gives:

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^1(T_{\mathrm{top}}, f_{T*}(\mathbb{G}_m)) \longrightarrow \mathrm{H}^1(X_{T,\mathrm{top}}, \mathbb{G}_m) \longrightarrow \mathrm{H}^0(T_{\mathrm{top}}, \mathrm{R}^1 f_{T*}(\mathbb{G}_m)) \\ \longrightarrow \mathrm{H}^2(T_{\mathrm{top}}, f_{T*}(\mathbb{G}_m)) \longrightarrow \mathrm{H}^2(X_{T,\mathrm{top}}, \mathbb{G}_m) \longrightarrow \cdots \end{aligned} \tag{4.1}$$

Notice that $\mathrm{Pic}_{X/S,\mathrm{top}}(T) = \mathrm{H}^0(T_{\mathrm{top}}, \mathrm{R}^1 f_{T*}(\mathbb{G}_m))$ from proposition 4.4. By proposition 4.2 we have $\mathrm{H}^1(X_{T,\mathrm{top}}, \mathbb{G}_m) = \mathrm{Pic}(X_T)$, and $\mathrm{H}^1(T_{\mathrm{top}}, f_{T*}(\mathbb{G}_m)) = \mathrm{Pic}(T)$. These two equalities are independent of the topology if $f_*(\mathcal{O}_X) = \mathcal{O}_S$ holds universally, from which we have $f_{T*}(\mathbb{G}_m) = \mathbb{G}_m$, or if f is proper [1]. The beginning of exact sequence (4.1) hence becomes

$$0 \longrightarrow \mathrm{Pic}(T) \longrightarrow \mathrm{Pic}(X \times_S T) \longrightarrow \mathrm{Pic}_{X/S,\mathrm{top}}(T) \tag{4.2}$$

[1] By Stein factorization ([Sta], 03GX), $f_T = g_T \circ f'_T$ such that $f'_{T,*} : X_T \to X'_T$ is proper with $f'_{T,*}(\mathcal{O}_{X_T}) = \mathcal{O}_{X'_T}$ and $g_T : X'_T \to T$ is a finite map which is *acyclic* for étale topology (thus for Zariski topology by descent), thus $H^1(T_{\mathrm{top}}, g_{T,*}(\mathcal{O}^*_{X'_T})) = H^1((X'_T)_{\mathrm{top}}, \mathcal{O}^*_{X'_T})$ (*cf.* [HP] Proposition 2.8 also), then $H^1(T_{\mathrm{top}}, f_{T,*}(\mathcal{O}^*_{X_T})) = H^1((X'_T)_{\mathrm{top}}, \mathcal{O}^*_{X'_T})$ for top = Zar, ét. The group $H^1((X'_T)_{\mathrm{top}}, \mathcal{O}^*_{X'_T})$ is independent of topology by proposition 4.2.

We are motivated to introduce another functor: $\operatorname{Pic}_{\text{quot}} : T \mapsto \operatorname{Pic}(X \times_S T)/\operatorname{Pic}(T)$. Hence (4.2) induces a natural inclusion $\operatorname{Pic}_{\text{quot}} \hookrightarrow \operatorname{Pic}_{X/S,\text{top}}$ of functors. If f has a section ϵ, then $f\epsilon = 1$ induces $\epsilon^* f^* = 1$. For each p, there is a left inverse of the map $\mathrm{H}^p(T, f_{T*}(\mathbb{G}_m) \xrightarrow{f^*} \mathrm{H}^p(X_T, \mathbb{G}_m)$, so f^* is injective. From exactness of (4.1), we have $\operatorname{Pic}_{\text{quot}} \xrightarrow{\sim} \operatorname{Pic}_{X/S,\text{top}}$.

We now close the general argument for the three top-cases. We compare the two exact sequences (4.1) for fppf- and ét- topology respectively, and use proposition 4.5 for $p = 2$. We see they have all five terms pairwise isomorphic except the middle ones. Then five lemma ensures that the middle terms are isomorphic too. In other words, we get that $\operatorname{Pic}_{X/S,\text{ét}} \cong \operatorname{Pic}_{X/S,\text{fppf}}$ whenever f is proper or $f_*(\mathcal{O}_X) = \mathcal{O}_S$ holds universally.

□

Corollary 4.7. *Assume $f_*(\mathcal{O}_X) = \mathcal{O}_S$ holds universally and f admits a section $\epsilon : S \to X$, $f\epsilon = 1$, the functor* $\operatorname{Pic}_{\text{quot}}$ *is a sheaf with respect to Zariski topology.*

5 Rigidifications and rigidificators

5.1 Rigidifications

Now we begin our introduction to rigidificators which will serve as a powerful tool for the comparison between $\operatorname{Pic}_{X/S,\text{fppf}}$, $\operatorname{Pic}_{\text{quot}}$, $\operatorname{Pic}_{X/S,\text{ét}}$, $\operatorname{Pic}_{X/S,\text{Zar}}$ and $\operatorname{Pic}_{X/S,\text{fpqc}}$ under certain assumptions. Recall that $f : X \to S$ is the structure map. For this subsection on rigidifications, we make the following assumption.

Assumption 5.1. $f_*(\mathcal{O}_X) = \mathcal{O}_S$ holds universally and f admits a section $\epsilon : S \to X$, $f\epsilon = 1$.

Definition 5.2. Let $T \in (\mathrm{Sch}/S)$ and $\mathcal{L}$ be a sheaf on X_T. A *rigidification along ϵ_T* or *ϵ_T-rigidification* is a choice of isomorphism $\alpha : \mathcal{O}_T \xrightarrow{\sim} \epsilon_T^*(\mathcal{L})$. The pair $(\mathcal{L}, \alpha)$ is called *an invertible sheaf rigidified along the section ϵ_T*.

Following [Raynaud] p30, we define *a homomorphism between* $(\mathcal{L}, \alpha)$ and $(\mathscr{M}, \beta)$ as a homomorphism $u : \mathcal{L} \to \mathscr{M}$ such that the following diagram commutes:

$$\begin{array}{ccc} (\mathcal{L}, \alpha) & \xrightarrow{\epsilon_T^*(u)} & (\mathscr{M}, \beta) \\ & {}_{\alpha}\nwarrow \quad \nearrow_{\beta} & \\ & \mathcal{O}_T & \end{array} \tag{5.1}$$

Given two pairs $(\mathcal{L}, \alpha)$, $(\mathscr{M}, \beta)$, we also define their sum as $(\mathcal{L} \otimes \mathscr{M}, \gamma)$, where γ is the composite morphism:

$$\gamma : \mathcal{O}_T \to \mathcal{O}_T \otimes_{\mathcal{O}_T} \mathcal{O}_T \xrightarrow{\alpha \otimes \beta} (\epsilon_T)^* \mathcal{L} \otimes_{\mathcal{O}_T} (\epsilon_T)^* \mathscr{M} \to (\epsilon_T)^* (\mathcal{L} \otimes \mathscr{M}) \tag{5.2}$$

Definition 5.3. Define the functor $(P_{X/S}, \epsilon) : (\mathrm{Sch}/S) \to (\mathrm{Sets})$ as follows: it associates each $T \in (\mathrm{Sch}/S)$ to the set $(P_{X/S}, \epsilon)(T)$ of isomorphism classes of invertible sheaves on X_T that are rigidified along the section $\epsilon_T : T \to X_T$.

We now justify the name rigidification:

Lemma 5.4. *Recall that we are under assumption 5.1. Let $T \in (\mathrm{Sch}/S)$, $\mathcal{L}$ be an invertible sheaf on X_T and α be an ϵ-rigidification. Then every automorphism of the pair $(\mathcal{L}, \alpha)$ is trivial.*

Proof. An automorphism of $(\mathcal{L}, \alpha)$ is an automorphism $v : \mathcal{L} \xrightarrow{\sim} \mathcal{L}$ such that $\epsilon_T^* v \circ \alpha : \mathcal{O}_T \xrightarrow{\sim} \epsilon_T^* \mathcal{L} \xrightarrow{\sim} \epsilon_T^* \mathcal{L}$ is equal to α, we must have $\epsilon_T^* v = 1$. Note that:

$$v \in \mathrm{Hom}(\mathcal{L}, \mathcal{L})^\times = \mathrm{H}^0(X_T, \mathscr{H}om(\mathcal{L}, \mathcal{L}))^\times = \mathrm{H}^0(X_T, \mathcal{O}_{X_T})^\times = \mathrm{H}^0(T, \mathcal{O}_T)^\times$$

where the last '=' is from $\mathcal{O}_T \xrightarrow{\sim} f_* \mathcal{O}_{X_T}$. $v \in \mathrm{H}^0(T, \mathcal{O}_T)^\times$ implies $v = f_T^* u$ for some $u \in \mathrm{H}^0(T, \mathcal{O}_T)^\times$. Applying $\epsilon_T^* f_T^* = 1$ and $\epsilon_T^* v = 1$, we have $u = 1$, thus $v = 1$. □

Lemma 5.5. *Recall that we are under the assumption 5.1. Define the functor $(P_{X/S}, \epsilon)(T)$ for $T \in (\mathrm{Sch}/S)$ by*

$(P_{X/S}, \epsilon)(T) := \{(\mathcal{L}, \alpha) \mid \mathcal{L}$ is an invertible sheaf on X_T, α is an ϵ_T-rigidification of $\mathcal{L}\}$.

Then there is a natural isomorphism $\rho : (P_{X/S}, \epsilon) \to \mathrm{Pic}_{\mathrm{quot}}$, such that ρ_T maps $(\mathcal{L}, \alpha)$ to $\mathcal{L}$ for each T.

Proof. Let $\mathscr{M}$ be an invertible sheaf on X_T that represents $\lambda \in \mathrm{Pic}_{\mathrm{quot}}(T)$. Set $\mathcal{L} := \mathscr{M} \otimes (f_T^* \epsilon_T^* \mathscr{M})^{-1}$, which also represents λ as $\epsilon_T^* \mathscr{M}$ is in $\mathrm{Pic}(T)$. As $\epsilon_T^* \mathcal{L} = \epsilon_T^* \mathscr{M} \otimes \epsilon_T^* f_T^* \epsilon_T^* \mathscr{M}^{-1}$, $\epsilon_T^* f_T^* = 1$ and the canonical isomorphism $\epsilon_T^* \mathscr{M} \otimes \epsilon_T^* \mathscr{M}^{-1} \xrightarrow{\sim} \mathcal{O}_T$ together give $\epsilon_T^* \mathcal{L} = \epsilon_T^* \mathscr{M} \otimes \epsilon_T^* \mathscr{M}^{-1} \xrightarrow{\sim} \mathcal{O}_T$ a ϵ_T-rigidification, thus ρ_T is surjective.

Let $(\mathcal{L}, \alpha)$ be an element in the kernel of ρ_T, then there is an invertible sheaf $\mathscr{N}$ on T and an isomorphism $\nu : \mathcal{L} \xrightarrow{\sim} f_T^* \mathscr{N}$. Set $\omega := \epsilon_T^* \nu \circ \alpha$. Then $\nu : (\mathcal{L}, \alpha) \xrightarrow{\sim} (f_T^* \mathscr{N}, \omega)$ and $f_T^* \omega : (\mathcal{O}_{X_T}, 1) \xrightarrow{\sim} (f_T^* \mathscr{N}, \omega)$, thus ρ_T is injective. □

Proposition 5.6. *Under the assumption 5.1, $(P_{X/S}, \epsilon)$ is a sheaf with respect to* fpqc*-topology.*

Proof. Consider the sequence:

$$(P_{X/S}, \epsilon)(T) \to (P_{X/S}, \epsilon)(T') \rightrightarrows (P_{X/S}, \epsilon)(T'')$$

where $T' \to T$ is an fpqc-morphism and $T'' = T' \times_T T'$, The first arrow is injective from [BLR] §6.1 theorem 4 which asserting that the fpqc-descent is effective. Fix $(\mathcal{L}', \alpha) \in (P_{X/S}, \epsilon)(T')$ whose image in $(P_{X/S}, \epsilon)(T'')$ coincide. We introduce the following notations: $\mathrm{pr}_i : T'' \to T'$, $p_i : X_{T''} \to X_{T'}$, α_i by pull-back of α via p_i, for $i = 1, 2$; $\mathrm{pr}_{ij} : T''' := T' \times_T T' \times_T T' \to T''$ and $p_{ij} : X_{T'''} \to X_{T''}$ and α_{ij} by pull-back of α via p_{ij},for $i < j$, $i, j \in \{1, 2, 3\}$.

The isomorphism $\omega : p_1^*(\mathcal{L}') \xrightarrow{\sim} p_2^*(\mathcal{L}')$ between the two pull-backs of $\mathcal{L}'$ to $X_{T''}$ is compatible with rigidifications because pairs $(p_i^*(\mathcal{L}'), \mathrm{pr}_i^*(\alpha))$ naturally satisfy $\epsilon_{T''}^* \omega \circ \alpha_1 = \alpha_2$ using $\epsilon_{T''}^* \circ p_i^* = \mathrm{pr}_i^* \circ \epsilon_{T'}^*$. Let ω_{ij} be the pull-back with respect to p_{ij}, then $\omega_{13}^{-1} \omega_{23} \omega_{12}$ is an automorphism of the pull-back of $(\mathcal{L}', \alpha)$ via the projection $p_{13}^* p_1^* : X_{T'''} \to X_{T'}$, by lemma 5.4, it must be trivial, so ω is a descent datum. The descent is effective by [BLR] §6.1 theorem 4, thus the above sequence is exact.

□

We conclude the comparison results we get so far under assumption 5.1. Lemma 5.5 tells us $(P_{X/S}, \epsilon) \xrightarrow{\sim} \mathrm{Pic}_{\mathrm{quot}}$. In the proof of proposition 4.6, we see $\mathrm{Pic}_{X/S,\mathrm{fppf}} \cong \mathrm{Pic}_{X/S,\text{ét}}$. The exact sequence (4.1) produces $\mathrm{Pic}_{\mathrm{quot}} \xrightarrow{\sim} \mathrm{Pic}_{X/S,\mathrm{Zar}}$ and $\mathrm{Pic}_{\mathrm{quot}} \xrightarrow{\sim} \mathrm{Pic}_{X/S,\mathrm{fppf}}$. Proposition 5.6 says $(P_{X/S}, \epsilon)$ is a fpqc-sheaf hence canonically isomorphic to $\mathrm{Pic}_{X/S,\mathrm{fpqc}}$.

In conclusion, we get $(P_{X/S}, \epsilon) \cong \mathrm{Pic}_{\mathrm{quot}} \cong \mathrm{Pic}_{X/S,\mathrm{Zar}} \cong \mathrm{Pic}_{X/S,\text{ét}} \cong \mathrm{Pic}_{X/S,\mathrm{fppf}} \cong \mathrm{Pic}_{X/S,\mathrm{fpqc}}$ if the assumption 5.1 is satisfied.

When $f_*(\mathcal{O}_X) = \mathcal{O}_S$ holds universally but we do not have the existence of a section $\epsilon : S \to X$, we need to generalize the notion of rigidifications to rigidificators.

5.2 Rigidificators

Definition 5.7. If $f : X \to S$ is proper, finitely generated, and flat, a subscheme $i_Y : Y \hookrightarrow X$, with Y being finite, flat and of finite presentation over S, is called a *rigidificator* of the relative Picard functor $\mathrm{Pic}_{X/S}$ (or of f) if the functor

$$(\mathrm{Sch}/S)^{\mathrm{opp}} \to (\mathrm{Sets}), \quad T \mapsto \Gamma(X_T, \mathcal{O}_{X_T})$$

is a subfunctor of the functor

$$(\mathrm{Sch}/S)^{\mathrm{opp}} \to (\mathrm{Sets}), \quad T \mapsto \Gamma(Y_T, \mathcal{O}_{Y_T}),$$

i.e. if the map $\Gamma(i_{Y_T}) : \Gamma(X_T, \mathcal{O}_{X_T}) \to \Gamma(Y_T, \mathcal{O}_{Y_T})$ is injective for any $T \in (\mathrm{Sch}/S)$.

If $f_*(\mathcal{O}_X) = \mathcal{O}_S$ holds universally, each section $\epsilon : S \to X$ of f defines a rigidificator of f, namely the closed subscheme $\epsilon(S) \subset X$.

Rigidificators exist, for example, in the following two cases (*cf.* [Raynaud] Prop.2.2.3):

(1) If the fibers of f do not have embedded components, then f admits a rigidificator locally over S with respect to the étale topology.

(2) If S is the spectrum of a discrete valuation ring, then f has a rigidificator.

Definition 5.8. Assume that $f : X \to S$ is proper, finitely generated, and flat. Let Y be a rigidificator of f. Then given $T \in (\mathrm{Sch}/S)$, an invertible sheaf $\mathcal{L}$ on X_T is called *rigidified along* Y_T (or Y_T*-rigidified*) if there is an isomorphism $\alpha : \mathcal{O}_{Y_T} \xrightarrow{\sim} i^*_{Y_T}(\mathcal{L})$, denoted by pair $(\mathcal{L}, \alpha)$.

Similarly to definition 5.2 we can define maps and sum between $(\mathcal{L}, \alpha)$ and $(\mathscr{M}, \beta)$ so that they form an abelian group. And similarly to lemma 5.4, we can prove that Y-rigidified invertible sheaves do not admit non-trivial automorphisms.

The functor $(\mathrm{Pic}_{X/S}, Y) : (\mathrm{Sch}/S)^{\mathrm{opp}} \to (\mathrm{Sets})$ defined by sending $T \in (\mathrm{Sch}/S)$ to the set of isomorphism classes of invertible sheaves on X_T which are Y-rigidified is actually a Zariski sheaf and, by descent, it is an fpqc-sheaf.

We can relate $(\mathrm{Pic}_{X/S}, Y)$ to $\mathrm{Pic}_{X/S}$ by projection to first factor just as we did for rigidifications case, but this is no longer an isomorphism in general. We need to know how much they differ.

We do some preparations on algebraic geometry by presenting the following definitions and theorem.

Definition 5.9. An $\mathcal{O}_X$-module $\mathscr{F}$ is called *locally of finite presentation* if it is locally isomorphic to the cokernel of a homomorphism $\mathcal{O}_X^m \to \mathcal{O}_X^n$ for some $m, n \in \mathbb{N}$.

Definition 5.10. Assume that $f : X \to S$ is proper and of finite presentation, and consider an $\mathcal{O}_X$-module $\mathscr{F}$ locally of finite presentation, which is flat over S. Then $\mathscr{F}$ is called *cohomologically flat over S in dimension* 0 if the formation of direct image $f_*(\mathscr{F})$ commutes with base change, *i.e.* for any base change diagram:

$$\begin{array}{ccc} X_T & \xrightarrow{u} & X \\ {\scriptstyle f_T}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow[v]{} & S \end{array} \tag{5.3}$$

the canonical map $v^* f_*(\mathscr{F}) \to f_{T,*} u^*(\mathscr{F})$ is an isomorphism. If this condition holds for $\mathscr{F} = \mathcal{O}_X$, we say that f is *cohomologically flat in dimension* 0.

Theorem 5.11. *Let $f : X \to S$ be a proper morphism which is finitely presented. Let $\mathscr{F}$ be an $\mathcal{O}_X$-module locally of finite presentation that is S-flat. Then there exists an $\mathcal{O}_S$-module $\mathscr{L}$ locally of finite presentation, unique up to canonical isomorphism, such that there is an isomorphism of functors*

$$f_*(\mathscr{F} \otimes_{\mathcal{O}_S} \mathscr{M}) \xrightarrow{\sim} \mathscr{H}om_{\mathcal{O}_S}(\mathscr{L}, \mathscr{M})$$

that is functorial in all quasi-coherent $\mathcal{O}_S$-modules $\mathscr{M}$. In particular, there is an isomorphism of functors

$$\Gamma(X, \mathscr{F} \otimes_{\mathcal{O}_S} \mathscr{M}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_S}(\mathscr{L}, \mathscr{M})$$

The $\mathcal{O}_S$-module $\mathscr{L}$ is locally free if and only if $\mathscr{F}$ is cohomologically flat over S in dimension 0. *In this case, $\mathscr{L}$ and $f_*(\mathscr{F})$ are dual to each other and $f_*(\mathscr{F})$ is locally free.*

Proof. See [EGA III.2] 7.7.6. ☐

If $f : X \to S$ is proper, finitely generated, and flat. Set $\mathscr{F} = \mathcal{O}_X$ and $\mathscr{M} = \mathcal{O}_T$ for $T \in (\mathrm{Sch}/S)$, the statement of theorem 5.11 becomes isomorphisms functorial in T:

$$\Gamma(X_T, \mathcal{O}_{X_T}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_S}(\mathscr{L}, \mathcal{O}_T) \xrightarrow{\sim} \mathrm{Hom}_S(T, V(\mathcal{L})) \tag{5.4}$$

where $V(\mathcal{L}) := \mathrm{Spec}(\mathcal{S}ym_{\mathcal{O}_S}(\mathcal{L}))$ (see [EGA II] definition I.7.8) and [EGA II] (I.7.13) gives the last isomorphism above. This can be read as: $V(\mathcal{L})$ represents the functor: $\Gamma(X_{(-)}, \mathcal{O}_{X_{(-)}})$. The scheme $V(\mathcal{L})$ is called *total space of the module $\mathcal{L}$*. It is called *locally free* if $\mathcal{L}$ is so as $\mathcal{O}_S$-module, which is equivalent to say that $V(\mathcal{L})$ is smooth over S.

Corollary 5.12. *If $f : X \to S$ is proper, finitely generated, and flat, let $\mathcal{L}$ be the $\mathcal{O}_S$-module associated to $f_*(\mathcal{O}_X)$ in the sense of Theorem 5.11. Then the functor*

$$(\mathrm{Sch}/S)^{\mathrm{opp}} \to (\mathrm{Sets}), \quad T \mapsto \Gamma(X_T, \mathcal{O}_{X_T})$$

is represented by the total space $V = V(\mathcal{L})$ of $\mathcal{L}$. Moreover, V is locally free if and only if $\mathcal{L}$ is cohomologically flat in dimension 0.

We have a characterization of rigidificators. For X, Y as in Definition 5.8, we use (5.4) to get their "global section after any base change" functor represented by V_X and V_Y respectively.

Proposition 5.13. *If $f : X \to S$ is proper, finitely generated, and flat, consider a subscheme $i_Y : Y \hookrightarrow X$ which is finite, flat and of finite presentation over S. Let $V_X \in (\mathrm{Sch}/S)$ (resp.V_Y) be representing the functor of global sections on X (resp.on Y). The following conditions are equivalent:*

(a) Y is a rigidificator of f.

(b) The morphism $V_X \to V_Y$ induced by the inclusion $i_Y : Y \hookrightarrow X$, is a closed immersion.

Proof. Let $\mathcal{L}$ (*resp.* $\mathcal{L}'$) denote the $\mathcal{O}_S$-module which is obtained from theorem 5.11 for $f : X \to S$ (*resp.* $f' = f \circ i_Y : Y \to S$). The for all $T \in (\mathrm{Sch}/S)$ such that $\mathcal{O}_T$ is a quasi-coherent $\mathcal{O}_S$-module, the inclusion $i_Y : Y \hookrightarrow X$ gives:

$$0 \to \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{L}, \mathcal{O}_T) \to \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{L}', \mathcal{O}_T) \tag{5.5}$$

Using identifications $\Gamma(X_T, \mathcal{O}_{X_T}) \cong \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{L}, \mathcal{O}_T)$ and $\Gamma(Y_T, \mathcal{O}_{Y_T}) \cong \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{L}', \mathcal{O}_T)$ from (5.4), we see that (5.5) is exact if and only if Y is a rigidificator of f. Now (5.5) corresponds to a sequence:

$$\mathcal{L}' \to \mathcal{L} \to 0 \tag{5.6}$$

of $\mathcal{O}_X$-modules that is exact if and only if (5.5) is exact for all T. Moreover, (5.6) yields a sequence between associated symmetric $\mathcal{O}_S$-algebras

$$\mathcal{S}ym_{\mathcal{O}_S}(\mathcal{L}') \to \mathcal{S}ym_{\mathcal{O}_S}(\mathcal{L}) \to 0 \tag{5.7}$$

which is exact if and only if it is exact of degree 1, if and only if (5.6) is exact. □

Let $V = V_X$, it represents a functor from schemes to rings, thus is a ring scheme.

Lemma 5.14. *If $f : X \to S$ is proper, finitely generated, and flat, then the subfunctor of units $T \mapsto \Gamma(X_T, \mathcal{O}^*_{X_T})$ is represented by an open subscheme $V^* \subset V$ which is a group scheme.*

Proof. See [BLR] §8.1 Lemma 10. □

Let $f : X \to S$ be proper, finitely generated, and flat. The canonical map $\mathcal{O}_S \to f_*(\mathcal{O}_X)$ defines a morphism $\mathbb{G}_a \to V_X$ that is a closed immersion, as can be seen by the proof of proposition 5.13. Restricting to the subscheme of units yields $\mathbb{G}_m \to V^*_X$, which is again closed.

We see that $f_*(\mathcal{O}_X) = \mathcal{O}_S$ holds universally if and only if $\mathbb{G}_a \to V_X$ is an isomorphism.

Let Y be a rigidificator of $f : X \to S$. The closed immersion $V_X \hookrightarrow V_Y$ gives rise to $V^*_X \hookrightarrow V^*_Y$, and there is a canonical map $V^*_Y \to (\mathrm{Pic}_{X/S}, Y)$ sending $a \in \Gamma(Y_T, \mathcal{O}^*_{Y_T})$ to $(\mathcal{O}_{X_T}, \alpha)$, where $\alpha : \mathcal{O}_{Y_T} \xrightarrow{\sim} i^*_{Y_T}(\mathcal{O}_{X_T}) \cong \mathcal{O}_{Y_T}$ is the multiplication by a. We hence have an exact sequence of sheaves in Zariski topology:

$$0 \to V^*_X \to V^*_Y \to (\mathrm{Pic}_{X/S}, Y) \to \mathrm{Pic}_{X/S} \to 0 \tag{5.8}$$

Moreover, it is even exact with respect to étale topology (see [Raynaud] 2.1.2 and 2.4.1).

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